TEACHING LOGIC: A NEW WAY OF CHECKING THE VALIDITY OF TRUTH FUNCTIONAL ARGUMENTS

IVAN L. LITTLE

What is proposed in this discussion is not a "new logic" but a "new" or different way of teaching students to check the validity of arguments in truth-functional logic. Since the procedure herein developed affords the skillful user a means for a rapid disclosure of the basic structure of an argument, we shall call it the "sketch cancellation" method (S-C). The term "sketch" refers to the elimination by simplification of unessential detail in logical statements and "cancellation" to the further elimination of elements by the "deductive" rule of the method. The simplification and cancellation rules will be fully clarified in subsequent discussion. These two rules will, in fact, replace the principles of modus ponens, modus tollens, hypothetical syllogism, or constructive and destructive dilemmas in purely S-C operations, because *they are in themselves adequate inference principles for the elimination of letters or letter combinations.* If, however, the solution of a problem requires an expansion of the given material, the rules of conjunction, addition, and absorption (which is actually a replacement rule) are adaptable to operations in S-C. It is theoretically possible that any of the usual textbook replacement rules might be required, but their use should constitute no significant problem. The method allows us to go to the heart of logical operations without being hampered by the cumbersome book-keeping rules required by the meticulous listing of steps. Even so, each move in the S-C game, if correctly done, is characterized by logical rigor. S-C users can engage in whatever book-keeping deemed to be necessary for clarity. This point will be brought up again in the final section.

The method will now be outlined:

1. *Conditionals to Disjunctions.*—All conditional propositions are transformed into their disjunctive equivalences, i.e., $p\supset q$ is rendered as ${\sim}p\vee q$. This transformation is required by the cancellation rule which operates on mixed or pure disjunctive syllogisms.

2. *Special Symbols in S-C.*—To facilitate operations the following symbols are used:

a. The arrow '$\rightarrow$' will replace the '$\supset$' in S-C operations. It is to be read as 'yields.'

b. The '$\leftrightarrow$' or bidirectional arrow replaces '$\equiv$' (the equivalence sign) in S-C. If there is no interest in the logical status of a yield, the '$\rightarrow$' can be used exclusively. But if instructors wish to stress each logical nuance, they should, then, respect the difference between the "uni" and "bi" directional arrows.

c. The bar '|' will be used for 'and,' usually only in the joinder of premisses for cancellation;[1] it never replaces the dot in conjunctive statements. By using the bar to join premisses, we reduce the number of parentheses and brackets required. For instance, $(p\vee q)\vee(r.s)\,|\,{\sim}r\vee{\sim}s$, which is to be read as "$(p\vee q)\vee(r.s)$ *with* ${\sim}r\vee{\sim}s$," is certainly easier to recognize as the joint assertion of *two* premisses than is $[(p\vee q)\vee(r.s)].({\sim}r\vee{\sim}s)$ which asserts one premiss.

The two unique rules in S-C now will be introduced and discussed. Also, we shall begin to number the examples for the sake of greater clarity.

3. *Simplification in S-C.*—Suppose we have the following argument form to be checked:

$$\begin{array}{ll}1.\ E\supset(F.G)\leftrightarrow{\sim}E\vee(F.G) & \\ 2.\ G\supset(H.I)\leftrightarrow{\sim}G\vee(H.I) & \\ 3.\ {\sim}H/\therefore{\sim}E & \end{array}\qquad(3.1)$$

We begin this check by transforming the conditional premisses, lines 1 and 2, into their disjunctive equivalences. But since the disjunctions themselves contain logically irrelevant material, they, in turn, require further simplification:

$$\begin{array}{l}1.\ {\sim}E\vee(F.G)\rightarrow{\sim}E\vee G\\ 2.\ {\sim}G\vee(H.I)\rightarrow{\sim}G\vee H\end{array}\qquad(3.2)$$

The normal procedure used to derive ${\sim}E\vee G$ and ${\sim}G\vee H$ from lines 1 and 2 in (3.2) is to distribute before simplifying. The rule of simplification in S-C, however, is interpreted to allow the yield of an element from a conjunction in any order, i.e., $p.q.r\rightarrow q,r$, or p or combinations such as $p.q$, $p.r$, etc. It is also expanded to cover disjunctive statements. Consequently, the rule for simplification in S-C (S-C Simp.) is *Letters can be dropped in any order from a conjunction; also single letters, or combinations of letters can be dropped from conjunctions in a disjunctive statement so long as at least one letter remains on each side of the symbol of disjunction.* An exception to this rule occurs in the case of an internally inconsistent conjunction as in $p\vee(q.{\sim}q)$ or in $p\vee[(q.{\sim}r).(q.r)]$. Each of these statements is equivalent to p; hence, in

$$p\vee(q.{\sim}q)\vee r\leftrightarrow p\vee r\qquad(3.3)$$

a conjunction is dropped from a disjunction. We can, therefore, put an addendum to S-C Simp.: *If, in a disjunctive statement a conjunction is internally inconsistent by being self-contradictory or self-contrary, it may be dropped in its entirety from the disjunction.* This addendum should be rarely required if we deal with logic rather than illogic.

4. *Cancellation (Can.).*—In (3.2) two simplified premisses ready for the operation of Can. in S-C have been derived from longer expressions. We shall now combine what we have done in (3.1) and (3.2) and complete the argument (note the labeling):

Given | *Immediate Derivations*

1. $E \supset (F.G)$ $\quad \leftrightarrow \sim Ev(F.G) \rightarrow \sim EvG$ $\qquad$ (4.1)
2. $G \supset (H.I)$ $\quad \leftrightarrow \sim Gv(H.I) \rightarrow \sim GvH$
3. $\sim H$

$\therefore \sim E$

Cancellations

4. $\sim Ev\not{G}(1) \,|\, \sim\not{G}vH(2) \rightarrow \sim EvH$
5. $\sim Ev\not{H}(4) \,|\, \sim\not{H}(3) \rightarrow \sim E$

The numbers in parentheses in lines 4 and 5 indicate the lines from which the expressions preceding them have been drawn.

Instructors and students may not, of course, actually use these labels in working problems since S-C aims at eliminating unnecessary work. Nevertheless, in (4.1) we have a simple model of the operations peculiar to S-C. Under the label "immediate derivations" the first yield results from the application of the simplification rule, and the second yield, that is, the simplified premisses which enter into the lines under "cancellations," is derived by S-C Simp., without the trivial moves afforded by the commutative and distributive laws. The actual deductive process occurs in lines 4 and 5 in (4.1). Strokes are shown on these lines for the letters that are cancelled. Line 5 shows that the conclusion has been validly derived.

The cancellation rule, which now must be stated, can be more easily formulated if we make the following distinctions:

a. Mixed Disjunctive Syllogism (M.D.S.):

pvq $\qquad$ In S-C: $pvq \,|\, \sim p \rightarrow q$ $\qquad$ (4.2)

$\sim p$

$\therefore q$

b. Pure Disjunctive Syllogism (P.D.S.):

$\sim pvq$ $\qquad$ In S-C: $\sim pvq \,|\, \sim qvr \rightarrow \sim pvr$ $\qquad$ (4.3)

$\sim qvr$

$\therefore \sim pvr$

So,

$$p.q \,|\, (\sim pv\sim q)vr \rightarrow r \qquad (4.4)$$

is an M.D.S. (containing only one disjunctive premiss), and

$$(p.q)v(rvs) \,|\, (p.\sim q)vt \rightarrow rvsvt \qquad (4.5)$$

is a pure disjunctive syllogism (containing two disjunctive premisses).

In (4.2) and (4.3) letters with opposite values are dropped and the remaining letter or letters asserted as final derivations. In (4.4) the compounds (p.q) and (~pv~q), being contradictories, are eliminated whereas (p.q) and (p.~q) in (4.5) are dropped because they are contraries. Hence S-C Can. may be stated as follows: *In mixed or pure disjunctive syllogisms:* (a) *a single letter is eliminated if it is positive in one premiss and negative in the other, and* (b) *a compound of letters appearing in the premisses as*

contradictories or as contraries in its two occurences can be eliminated. The conclusion of an M.D.S. consists of the uneliminated part of the disjunctive premiss, and the conclusion of a P.D.S. contains the uncancelled parts of both premisses stated as a single disjunction.

The following S-C argument is valid:

$$p.\sim q \,|\, (\sim p.\sim q)vr \rightarrow r \qquad (4.6)$$

because (p.~q) and (~p.~q) are contraries, but (4.7) is invalid:

$$\sim pv\sim q \,|\, (pvq)vr \rightarrow r \qquad (4.7)$$

Since (~pv~q) and (pvq) are subcontraries they can be true together. Only those statements which cannot be true together are "cancellable."

5. *Contradiction and Contrariety.*—Hence a clear understanding of contradiction and contrariety is crucial to the successful mastery of S-C. An expanded set of De Morgan's rules expressing propositional opposition rather than equivalence reveals the contradiction obtaining between propositional pairs, e.g., p.q is contradicted by ~pv~q, p.~q by ~pvq, ~p.q by pv~q, and ~p.~q, by pvq. Thus, *conjunctive statements are contradicted by disjunctions containing the same letters or compounds of letters with opposite truth values.* Moreover, the relationship which these four conjunctions bear to each other is that of contrariety, they cannot be true but can be false together. The four disjunctions are subcontraries; they can be true but not false together. Each conjunction is a superaltern to all of the disjunctions in the set, except to its contradictory.

Not only are contradiction and contrariety requisite to cancellation, but contradiction is inherent in the transformation of conditionals into disjunctions. It is, therefore, essential to learn to recognize the contradictories of the antecedents of complex conditional statements.

Say that a given premiss is:

$$[(p.\sim q)v(\sim rvs)] \supset (t.u) \qquad (5.1)$$

We tend to begin its transformation into a disjunction by this procedure:

$$\sim[(p.\sim q)v(\sim rvs)]\, v(t.u) \qquad (5.2)$$

But, (5.2) is of no use in S-C. So, knowing what a contradiction is, we should be able to go immediately (perhaps not at once, but with practice) from (5.1) to

$$[(\sim pvq).(r.\sim s)]\, v(t.u) \qquad (5.3)$$

since probably all we want from it is ~svu or some similarly simple ensemble.

6. *Working problems in S-C.*—In the three problems analyzed below, two of which are relatively difficult to prove with the ordinary techniques used in truth-functional logic, the range of flexibility in S-C is stressed.

1. $[p \supset q).(q \supset p] \supset r \leftrightarrow [(p.\sim q)v(q.\sim p)]\, vr \rightarrow pvqvr$ $\qquad$ (6.1)
2. $r \supset [(p \supset q).(q \supset p)] \rightarrow \sim rv(\sim qvp)$

3. $q \equiv r \to a. \sim q \vee r$
$\quad\to b. \sim r \vee q$
$\quad\therefore p$
4. $p \vee q \vee r(1) \mid \sim q \vee r(3a) \to p \vee (r \vee r) \leftrightarrow p \vee r^*$
5. $\sim r \vee (\sim q \vee p)(2) \mid \sim r \vee q(3b) \to (\sim r \vee \sim r) \vee p \leftrightarrow \sim r \vee p^*$
6. $p \vee r(4) \mid \sim r \vee p(5) \to p \vee p \leftrightarrow p^*$
*Taut.

A similar problem, but one that will require the S-C version of the absorption rule (Abs.) follows. Since the absorption rule is $(p \supset q) \equiv [p \supset (p.q)]$, in S-C it is $(\sim p \vee q) \leftrightarrow [\sim p \vee (p.q)]$.

(6.2)

1. $(p \supset q) \supset (r \supset s) \leftrightarrow (p.\sim q) \vee (\sim r \vee s) \to \sim q \vee \sim r \vee s$
2. $p \supset r \qquad \leftrightarrow \sim p \vee r$
$\quad\therefore (p.q) \supset (q.s)$
3. $\sim p \vee r(2) \mid \sim q \vee \sim r \vee s(1) \to \sim p \vee (\sim q \vee s)$ (Can.)
$\leftrightarrow \sim p \vee [\sim q \vee (q.s)]$ (Abs.)
$\leftrightarrow (\sim p \vee \sim q) \vee (q.s)$ (Assoc.)
$\leftrightarrow \sim(\sim p \vee \sim q) \supset (q.s)$ (Impl.)
$\leftrightarrow (p.q) \supset (q.s)$ (DeM.)

Reference to the rules of cancellation, absorption, association, implication, and DeMorgan in line 3 of (6.2) shows how authority can be cited on a line with whatever detail required or considered suitable. As indicated earlier, authority references may be omitted if the aim is speed and simplicity and if the student understands what is being done. In (6.1) the use of the tautology rule is indicated for lines 4, 5, and 6. A similar relaxation in the use of parentheses and brackets is appropriate except in groupings pertinent to cancellation of compounds or required by the very nature of the given problem. For instance, the conclusion of (6.2) necessitates the use of parentheses to group $(\sim p \vee \sim q)$ for its transformation by Impl. and DeM. into (p.q), the antecedent of the conditional conclusion.

A final remark about cancellation may not be inappropriate. Letters may be cancelled in one context and reused in another. We may have:

(6.3)

1. $p \supset q \leftrightarrow \sim p \vee q$
2. $q \supset r \leftrightarrow \sim q \vee r$
3. $q \supset s \leftrightarrow \sim q \vee s$
$\quad\therefore p \supset (r.s)$
4. $\sim p \vee q(1) \mid \sim q \vee r(2) \to \sim p \vee r$
5. $\sim p \vee q(1) \mid \sim q \vee s(3) \to \sim p \vee s$
6. $\sim p \vee r(4) \mid \sim p \vee s(5) \to \sim p \vee (r.s)$ (Dist.) $\leftrightarrow p \supset (r.s)$

The "moves" in S-C operations in (6.3) should require no further comment. Nor, for that matter, should further elucidation of the method be needed. Application of S-C rules to deductive manipulations in quantification logic should be obvious.

NOTE

[1] Sometimes, of course, premisses may be joined for purposes other than cancellation. Line 6 in (6.3), which uses the distribution rule, is a case in point.

Texas Tech University